Three Cube Roots of Unity & Four Fourth Roots of Unity

Three Cube Roots & Four Fourth Roots of a Number

Exercise 4.4

1. Find the three cube roots of: 8, -8, 27, -27, and 64

(I) 8

Solution:

Let \( x \) be a cube root of 8

\[ x = \sqrt[3]{8} = (8)^{\frac{1}{3}} \]

\[ x^3 = ((8)^{\frac{1}{3}})^3 \]

\[ x^3 = 8 \]

\[ x^3 - 8 = 0 \]

\[ (x)^3 - (2)^3 = 0 \]

\[ (x - 2)(x^2 + 2x + 4) = 0 \]

\[ x - 2 = 0 \]

\[ x = 2 \]

\[ x^2 + 2x + 4 = 0 \]

\[ x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)} \]

\[ x = \frac{-2 \pm \sqrt{-12}}{2} \]

\[ x = \frac{-2 \pm \sqrt{-4(3)}}{2} \]

\[ x = \frac{-2 \pm 2\sqrt{3}}{2} \]

\[ x = 2(-1 \pm \frac{\sqrt{3}i}{2}) \]  \( \Rightarrow \)  \[ x = 2(\frac{-1 \pm \sqrt{3}i}{2}) \]

\[ x = 2w \]

\[ x = 2w^2 \]

\[ Q \ w = \frac{-1 + \sqrt{3}i}{2} \]

\[ Q \ w^2 = \frac{-1 - \sqrt{3}i}{2} \]

Hence, three cube roots of 8 are: \{2, 2w, 2w^2\}
(II) -8
Solution:
Let \( x \) be a cube root of -8
\[ \therefore x = \sqrt[3]{-8} = (-8)^{\frac{1}{3}} \]
\[ x^3 = ((-8)^{\frac{1}{3}})^3 \]
\[ x^3 = -8 \]
\[ x^3 + 8 = 0 \]
\[ (x)^3 + (2)^3 = 0 \]
\[ (x + 2)(x^2 - 2x + 4) = 0 \]
\[ x + 2 = 0 \quad \text{or} \quad x^2 - 2x + 4 = 0 \]
\[ x = -2 \quad \text{or} \quad x = -2 \pm \sqrt{-12} \]
\[ x = -2 \quad \text{or} \quad x = 2 \pm \sqrt{-12} \]
\[ x = 2 \pm \sqrt{-12} \]
\[ x = 2 \pm \sqrt{(4)(3)} \]
\[ x = 2 \pm \sqrt{-3} \]
\[ x = \frac{2(1 \pm \sqrt{3}i)}{2} \quad \Rightarrow \quad x = 2\left(\frac{1 \pm \sqrt{3}i}{2}\right) \]
\[ x = 2\left(\frac{1+\sqrt{3}i}{2}\right) \quad x = 2\left(\frac{1-\sqrt{3}i}{2}\right) \]
\[ x = 2\left(-\frac{1+\sqrt{3}i}{2}\right) \quad x = 2\left(-\frac{1-\sqrt{3}i}{2}\right) \]
\[ x = -2w^2 \quad x = -2w \]
\[ Q \quad w = \frac{-1 + \sqrt{3}i}{2} \]
\[ Q \quad w^2 = \frac{-1 - \sqrt{3}i}{2} \]

Hence, three cube roots of -8 are: \{-2, -2w, -2w^2\}

(III) 27
Solution:
Let \( x \) be a cube root of 27
\[ \therefore x = \sqrt[3]{27} = (27)^{\frac{1}{3}} \]
\[ x^3 = ((27)^{\frac{1}{3}})^3 \]
\[ x^3 = 27 \]
\[ x^3 - 27 = 0 \]
(x)^3 - (3)^3 = 0
(x - 3)(x^2 + 3x + 9) = 0
x - 3 = 0
x = 3

x^2 + 3x + 9 = 0
x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}
= \frac{-3 \pm \sqrt{-27}}{2}
= \frac{-3 \pm 3\sqrt{-3}}{2}
= \frac{3(-1 \pm \sqrt{-3})}{2}

\Rightarrow x = \frac{3(-1 \pm \sqrt{3}i)}{2}

Hence, three cube roots of 27 are: \{3, 3w, 3w^2\}

(IV) -27
Solution:
Let x be a cube root of -27
\therefore x = \sqrt[3]{-27} = (-27)^\frac{1}{3}

x^3 = ((-27)^\frac{1}{3})^3
x^3 = -27
x^3 + 27 = 0
(x)^3 + (3)^3 = 0
(x + 3)(x^2 - 3x + 9) = 0
x + 3 = 0
x = -3

x^2 - 3x + 9 = 0
x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}
= \frac{3 \pm \sqrt{-27}}{2}
= \frac{3 \pm \sqrt{-9(3)}}{2}
\[ x = \frac{3 \pm \sqrt[3]{-3}}{2} \]
\[ x = \frac{3(1 \pm \sqrt[3]{-3})}{2} \]
\[ x = \frac{3(1 \pm \sqrt[3]{3}i)}{2} \quad \Rightarrow \quad x = \frac{1 \pm \sqrt[3]{3}i}{2} \]
\[ x = \frac{3(1 \pm \sqrt{3}i)}{2} \]
\[ x = \frac{3(1 \pm \sqrt{3}i)}{2} \quad x = \frac{3(1 - \sqrt[3]{3}i)}{2} \]
\[ x = -\frac{3(1 - \sqrt{3}i)}{2} \quad x = -\frac{1 \pm \sqrt[3]{3}i}{2} \]
\[ x = -3w^2 \quad x = -3w \]

Hence, three cube roots of -27 are: \{-3, -3w, -3w^2\}

(V) 64
Solution:
Let \( x \) be a cube root of 64
\[ : x = \sqrt[3]{64} = (64)^{\frac{1}{3}} \]
\[ x^3 = ((64)^{\frac{1}{3}})^3 \]
\[ x^3 = 64 \]
\[ x^3 - 64 = 0 \]
\[ (x - 4)(x^2 + 4x + 16) = 0 \]
\[ x - 4 = 0 \quad x^2 + 4x + 16 = 0 \]
\[ x = 4 \]
\[ x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(16)}}{2(1)} \]
\[ x = \frac{-4 \pm \sqrt{-48}}{2} \]
\[ x = \frac{-4 \pm \sqrt{-16(3)}}{2} \]
\[ x = \frac{-4 \pm 4\sqrt{3}}{2} \]
\[ x = \frac{4(-1 \pm \sqrt{3}i)}{2} \quad \Rightarrow \quad x = 4\left(\frac{-1 \pm \sqrt{3}i}{2}\right) \]
\[ x = 4w \quad x = 4w^2 \]

Hence, three cube roots of 64 are: \{4, 4w, 4w^2\}
2. Evaluate:
   (i) \((1 + w - w^2)^8\)
   (ii) \(w^{28} + w^{29} + 1\)
   (iii) \((1 + w - w^2)(1 - w + w^2)\)
   (iv) \(\left(\frac{-1 + \sqrt{3}}{2}\right)^9 + \left(\frac{-1 - \sqrt{3}}{2}\right)^7\)
   (v) \((-1 + \sqrt{3})^5 + (-1 - \sqrt{3})^5\)

(i) \((1 + w - w^2)^8\)
Solution:
\[(1 + w - w^2)^8 = (1 + w + w^2 - w - w^2)^8\]
\[= (1 + w + w^2 - 2w^2)^8\]
\[= (0 - 2w^2)^8\]
\[Q 1 + w + w^2 = 0\]
\[= 256w^16\]
\[= 256(w^8)^2w\]
\[= 256(1)^2w\]
\[= 256w\]

(ii) \(w^{28} + w^{29} + 1\)
Solution:
\[w^{28} + w^{29} + 1 = (w^3)^9w + (w^3)^9w^2 + 1\]
\[= (1)^9w + (1)^9w^2 + 1\]
\[= w + w^2 + 1\]
\[= 0\]
\[Q w + w^2 + 1 = 1 + w + w^2 = 0\]

(iii) \((1 + w - w^2)(1 - w + w^2)\)
Solution:
\[(1 + w - w^2)(1 - w + w^2)\]
Add and subtract \(w^2\) in the 1st bracket, and add and subtract \(w\) in the 2nd bracket.
\[(1 + w - w^2)(1 - w + w^2) = (1 + w + w^2 - w^2 - w^2)(1 + w - w + w^2)\]
\[= (1 + w + w^2 - 2w^2)(1 + w + w^2 - 2w)\]
\[= (0 - 2w^2)(0 - 2w)\]
\[= (-2w^2)(-2w)\]
\[= 4w^3\]
\[= 4\]

(iv) \(\left(\frac{-1 + \sqrt{3}}{2}\right)^9 + \left(\frac{-1 - \sqrt{3}}{2}\right)^7\)
Solution:
\[(\frac{-1+\sqrt{3}}{2})^9 + (\frac{-1-\sqrt{3}}{2})^9 = (w)^9 + (w^2)^9\]
\[Q w = \frac{-1+\sqrt{3}}{2}, \quad w^2 = \frac{-1-\sqrt{3}}{2}\]

If \[(\frac{-1+\sqrt{3}}{2})^7 + (\frac{-1-\sqrt{3}}{2})^7\]
Then solution:
\[(\frac{-1+\sqrt{3}}{2})^7 + (\frac{-1-\sqrt{3}}{2})^7 = (w)^7 + (w^2)^7\]
\[= (w^3)^7 + w^7\]
\[= (w^3)^7 + (w^3)^4 w^2\]
\[= w^7 + w^2\]
\[= -1\]

(v) \[(\frac{-1+\sqrt{3}}{2})^5 + (\frac{-1-\sqrt{3}}{2})^5\]
Solution:
Multiply and divide by “2” with both the brackets.
\[(-1+\sqrt{3})^5 + (-1-\sqrt{3})^5 = (2\times\frac{-1+\sqrt{3}}{2})^5 + (2\times\frac{-1-\sqrt{3}}{2})^5\]
\[= (2\times w)^5 + (2\times w^2)^5\]
\[= 32w^5 + 32w^{10}\]
\[= 32w^5w^2 + 32w^3w^2\]
\[= 32(1)w^2 + 32(1)w\]
\[= 32w^2 + 32w\]
\[= 32(w^2 + w)\]
\[= 32(-1)\]
\[= -32\]

3. Show that:
(i) \[x^3 - y^3 = (x - y)(x - wy)(x - w^2 y)\]
Solution:
\[x^3 - y^3 = (x - y)(x - wy)(x - w^2 y)\]
\[x^3 - y^3 = (x^2 - wxy - xy + w^2 y^2)(x - w^2 y)\]
\[x^3 - y^3 = (x^3 - w^2 x^2 y - w^2 y + w^3 y^2 - x^3 y + w^2 xy^2 + wxy^2 - w^3 y^3)\]
Rearranging we have,
\[ x^3 - y^3 = (x^3 - w^2 x^2 y - wx^2 y - x^2 y + w^2 x y^2 + w x y^2 - w^2 y^3) \]
\[ x^3 - y^3 = x^3 - x^2 y (w^2 + w + 1) + x y^2 (w^3 + w^2 + w) - w^3 y^3 \]
\[ x^3 - y^3 = x^3 - x^2 y (w^2 + w + 1) + w x y^2 (w^2 + w + 1) - w^3 y^3 \]
\[ x^3 - y^3 = x^3 - w^3 y^3 \]
\[ x^3 - y^3 = x^3 - y^3 \quad \text{Proved.} \]

(ii) \[ x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + wy + w^2 z)(x + w^2 y + wz) \]

Solution:
\[ x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + wy + w^2 z)(x + w^2 y + wz) \]
Take L.H.S
\[ = (x + y + z)(x + wy + w^2 z)(x + w^2 y + wz) \]
\[ = (x + y + z)(x^2 + w^2 xy + wxz + wxy + w^2 y^2 + w^2 yz + w^2xz + w^2 yz + w^2 z^2) \]
\[ = (x + y + z)(x^2 + x y (w^2 + w) + xz (w + w^2) + yz (w^2 + w) + w^3 y^2) \]
\[ = (x + y + z)(x^2 + x ( -1) + xz ( -1) + yz ( -1) + (1)y^2) \]
\[ = (x + y + z)(x^2 - xy - xz - yz + y^2) \]
\[ = x^3 - x^2 y - x^2 z - xyz + xy^2 + x^2 y - xy^2 - xyz - y^2 z + y^2 + x^2 z - xyz - xz^2 - yz^2 + y^2 z \]
\[ = x^3 + y^3 + z^3 - 3xyz \quad \text{Proved.} \]

(iii) \[ (1+w)(1+w^2)(1+w^4)(1+w^8) \quad \ldots \quad 2n \text{ factors} = 1 \]

Solution:
\[ (1+w)(1+w^2)(1+w^4)(1+w^8) \quad \ldots \quad 2n \text{ factors} \]
\[ (1+w)(1+w^2)(1+w)(1+w^2) \quad \ldots \quad 2n \text{ factors} \]
\[ (1+w^2 + w + w^4)(1+w + w^2) \quad \ldots \quad 2n \text{ factors} \]
\[ (1+w^2 + w + 1)(1+w^2 + w + 1) \quad \ldots \quad 2n \text{ factors} \]
\[ (0+1)(0+1) \quad \ldots \quad 2n \text{ factors} \]
\[ (1) \quad \ldots \quad 2n \text{ factors} \]
\[ (1)^{2n} \]
\[ 1 \quad \text{Proved.} \]

4. If \( w \) is a root of \( x^2 + x + 1 = 0 \), show that its other root is \( w^2 \), and prove that \( w^3 = 1 \).

Solution:
If \( w \) is the root of the equation \( x^2 + x + 1 = 0 \) then,
\[ w^2 + w + 1 = 0 \quad \rightarrow \text{Equation (A)} \]
Now, to prove that its other root is \( w^2 \), Replace \( x \) by \( w^2 \),
\[ x^2 + x + 1 = 0 \quad \Rightarrow \quad (w^2)^2 + w^2 + 1 = 0 \]
Add and subtract \( w^2 \).
\[ (w^2)^2 + w^2 + w^2 + 1 - w^2 = 0 \]
\[ (w^2)^2 + 2w^2 + 1 - w^2 = 0 \]
\[(w^2 + 1)^2 - w^2 = 0\]
\[(w^2 + 1 + w)(w^2 + 1 - w) = 0\]
\[Q\quad a^2 - b^2 = (a + b)(a - b)\]

From Equation (A) \(w^2 + 1 + w = 0\), therefore
\[(0)(w^2 + 1 - w) = 0\]
\[\Rightarrow (w^2)^2 + w^2 + 1 = 0 \rightarrow \text{Equation (B)}\]

Hence, \(w^2\) is also a root of \(x^2 + x + 1 = 0\).

Now, to prove \(w^3 = 1\) subtract Equation (A) from B.
\[(w^3)^2 + w^2 + 1 = 0\]
\[w^2 \pm w \pm 1 = 0\]
\[w^4 - w = 0\]
\[w(w^3 - 1) = 0\]
As, \(w \neq 0\)
\[w^3 - 1 = 0\]
\[w^3 = 1\]
Proved.

5. Prove that complex cube roots of \(-1\) are \(\frac{1 + \sqrt{3}i}{2}\) and \(\frac{1 - \sqrt{3}i}{2}\); and hence prove that
\[\left(\frac{1 + \sqrt{3}i}{2}\right)^3 + \left(\frac{1 - \sqrt{3}i}{2}\right)^3 = -2\]

Solution:
Let \(x\) be the cube root of \(-1\), therefore
\[x = \sqrt[3]{-1} = (-1)^\frac{1}{3}\]
\[x^3 = -1\]
\[x^3 + 1 = 0\]
\[(x + 1)(x^2 - x + 1) = 0\]
\[x + 1 = 0\]
\[x^2 - x + 1 = 0\]
\[x = -1\]
\[x = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}\]
\[x = \frac{1 + \sqrt{3}i}{2}\]
\[x = \frac{1 - \sqrt{3}i}{2}\]
\[x = \frac{1 + \sqrt{3}i}{2}\]
\[x = \frac{1 - \sqrt{3}i}{2}\]

Hence, three cube roots of \(-1\) are:\(-1, \frac{1 + \sqrt{3}i}{2}, \frac{1 - \sqrt{3}i}{2}\)
Now, we have to prove, \( \left( \frac{1 + \sqrt{-3}}{2} \right)^{9} + \left( \frac{1 - \sqrt{-3}}{2} \right)^{9} = -2 \) \[ \rightarrow \text{Equation (C)} \]

As we know that: \( w = \frac{-1 + \sqrt{-3}}{2} \) and \( w^2 = \frac{-1 - \sqrt{-3}}{2} \)

\(-w = \frac{1 - \sqrt{-3}}{2} \) and \( -w^2 = \frac{1 + \sqrt{-3}}{2} \)

Hence, equation (C) becomes:
\((-w^2)^{9} + (-w)^{9} = -2\)

Take R.H.S
\((-w^2)^{9} + (-w)^{9} = -w^{18} - w^{9} = -(w^3)^6 - (w^3)^3 = -(1)^6 - (1)^3 = -2 \) \[ \text{Proved.} \]

6. If \( w \) is a cube root of unity, form an equation whose roots are \( 2w \) and \( 2w^2 \).

Solution:
To form an equation, whose roots are \( 2w \) and \( 2w^2 \), we have
\((x - 2w)(x - 2w^2) = 0\)
\[ Q \] If \( 2w \) is a root \( \Rightarrow x = 2w \Rightarrow x - 2w = 0 \)
Is \( 2w^2 \) is a root \( \Rightarrow x = 2w^2 \Rightarrow x - 2w^2 = 0 \)

\( x^2 - 2w^2x - 2wx + 4w^3 = 0 \)
\( x^2 - 2x(w^2 + w) + 4w^3 = 0 \)
\( x^2 - 2x(-1) + 4(1) = 0 \) \[ Q \] \( w^6 + w = -1, w^3 = 1 \)
\( x^2 + 2x + 4 = 0 \) \[ \text{Answer} \]

7. Find four fourth roots of: 16, 81, and 625
(i) 16

Solution:
Let \( x \) be a fourth root of 16, therefore,
\( x = \sqrt[4]{16} \)
\( x = 16^{\frac{1}{4}} \)

\( (x)^4 = \left( 16^{\frac{1}{4}} \right)^4 \)
\( x^4 = 16 \)
\( x^4 - 16 = 0 \)
\( (x^2)^2 - (4)^2 = 0 \) \[ Q \] \( a^2 - b^2 = (a + b)(a - b) \)
\( (x^2 + 4)(x^2 - 4) = 0 \)
\( x^2 + 4 = 0 \)
\( x^2 - 4 = 0 \)
\( x^2 = -4 \)
\( x^2 = 4 \)
\( x = \pm \sqrt{-4} \)
\( x = \pm \sqrt{4} \)
\[ x = \pm \sqrt[4]{4i} \quad x = \pm 2i \]

Hence four fourth roots of 16 are: \( 2, -2, 2i, -2i \)

(ii) 81
Solution:
Let \( x \) be a fourth root of 81, therefore,
\[ x = \sqrt[4]{81} \]
\[ x = 81^{\frac{1}{4}} \]
\[ (x)^4 = \left( 81^{\frac{1}{4}} \right)^4 \]
\[ x^4 = 81 \]
\[ x^4 - 81 = 0 \]
\[ (x^2)^2 - (9)^2 = 0 \]
\[ (x^2 + 9)(x^2 - 9) = 0 \]
\[ Q \ a^2 - b^2 = (a + b)(a - b) \]
\[ x^2 + 9 = 0 \quad x^2 - 9 = 0 \]
\[ x^2 = -9 \quad x^2 = 9 \]
\[ x = \sqrt{-9} \quad x = \sqrt{9} \]
\[ x = \pm 3i \quad x = \pm 3 \]
Hence, four fourth roots of 81 are: \( 3, -3, 3i, -3i \)

(iii) 625
Solution:
Let \( x \) be a fourth root of 625, therefore,
\[ x = \sqrt[4]{625} \]
\[ x = 625^{\frac{1}{4}} \]
\[ (x)^4 = \left( 625^{\frac{1}{4}} \right)^4 \]
\[ x^4 - 625 = 0 \]
\[ (x^2)^2 - (25)^2 = 0 \]
\[ (x^2 + 25)(x^2 - 25) = 0 \]
\[ Q \ a^2 - b^2 = (a + b)(a - b) \]
\[ x^2 + 25 = 0 \quad x^2 - 25 = 0 \]
\[ x^2 = -25 \quad x^2 = 25 \]
\[ x = \sqrt{-25} \quad x = \sqrt{25} \]
\[ x = \pm 5i \quad x = \pm 5 \]
Hence, four fourth roots of 625 are: 5, −5, 5i, −5i

8. Solve the following equations:
   (i) \(2x^4 - 32 = 0\)
   Solution:
   \[2x^4 = 32\]
   \[2(x^4 - 16) = 0\]
   \[x^4 - 16 = 0\]
   \[(x^2)^2 - (4)^2 = 0\]
   \[(x^2 + 4)(x^2 - 4) = 0\]
   \[Q (a^2 - b^2) = (a + b)(a - b)\]
   \[x^2 + 4 = 0\]
   \[x^2 - 4 = 0\]
   \[x^2 = -4\]
   \[x = \sqrt{-4} = 2i\]
   \[x = \sqrt{4i}\]
   \[x = \pm 2i\]
   Hence, solution set = \(\{2, -2, 2i, -2i\}\)

   (ii) \(3y^5 - 243y = 0\)
   Solution:
   \[3y^5 - 243y = 0\]
   \[3y(y^4 - 81) = 0\]
   \[3y = 0\]
   \[y^4 = 81 = 0\]
   \[y = 0\]
   \[(y^2)^2 - (9)^2 = 0\]
   \[(y^2 + 9)(y^2 - 9) = 0\]
   \[Q (a^2 - b^2) = (a + b)(a - b)\]
   \[y^2 + 9 = 0\]
   \[y^2 - 9 = 0\]
   \[y^2 = -9\]
   \[y = \pm \sqrt{-9} = \pm 3i\]
   \[y^2 = 9\]
   \[y = \pm \sqrt{9} = \pm 3\]
   Hence, solution set = \(\{0, 3, -3i, -3i\}\)

   (iii) \(x^3 + x^2 + x + 1 = 0\)
   Solution:
   \[x^3 + x^2 + x + 1 = 0\]
   \[x^2(x + 1) + 1(x + 1) = 0\]
   \[(x + 1)(x^2 + 1) = 0\]
   \[x + 1 = 0\]
   \[x^2 + 1 = 0\]
   \[x = -1\]
   \[x^2 = -1\]
   \[x = \pm \sqrt{-1} = \pm i\]
   \[Q \ i^2 = -1\]
   \[x = \pm \sqrt{i^2}\]
\[ x = \pm i \]
Hence, solution set = \{-1, i, -i\}

(iii) \( 5x^5 - 5x = 0 \)
Solution:
\[
5x^5 - 5x = 0 \\
5x(x^4 - 1) = 0
\]
\[
x = 0 \\
x^4 - 1 = 0
\]
\[
(x^2 + 1)(x^2 - 1) = 0
\]
\[
x^2 + 1 = 0 \\
x^2 = -1
\]
\[
x = \pm i
\]
\[
x^2 - 1 = 0 \\
x^2 = 1
\]
\[
x = \pm 1
\]
Hence, solution set = \{0, 1, -1, i, -i\}